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STRATEGIC PROFIT SHARING BETWEEN FIRMS: THE BERTRAND MODEL *

Roberts Waddle [1]

Abstract

The present paper first considers two firms in a homogeneous market competing in a two-stage game. Using a particular strategy, it shows that firms may be able *to set prices above the marginal costs and thus get positive profits*. This remarkable result is *robust to the number of firms and to cost asymmetries.*

Furthermore and more importantly, when firms' costs are different, firms obtain positive profits even though they set prices at the highest marginal cost.

Key Words: Profit sharing, Oligopoly, Bertrand paradox, Competition.

* **Acknowledgements:** We are grateful to our supervisor José Luis Ferreira Garcia for his numerous suggestions and comments. Nevertheless, all remained errors are ours.

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Strategic Profit Sharing Between Firms: The Bertrand Model[1]

Roberts Waddle[2]

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Preliminary- Comments welcome!
(*please, do not circulate*)

[1]I'm very grateful to my supervisor José Luis Ferreira for his numerous helpful suggestions. Nevertheless, all remained errors are my own.

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Abstract

The present paper first considers two firms in an homogeneous market competing in a two-stage game. Using a particular strategy, it shows that firms may be able *to set prices above the marginal costs* and thus get positive profits. This remarkable result is *robust to the number of firms and to cost asymmetries.*

Furthermore and more importantly, when firms' costs are different, *all firms* obtain positive profits even though they set prices at the highest marginal cost.

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JEL Classification: C72, D21, L13.

"There is more happiness in giving than there is in receiving"
New Testament

1 Introduction

The Bertrand (1883) paradox has always fascinated and still fascinate economists. Some have strongly criticized it pointing out its lack of realism. For instance, they think that it could be improved by relaxing some of its crucial assumptions like the timing of the game or the perfect substitutability of products. Others have attempted to find out a solution to it. For example, Edgeworth (1897) solved it by introducing the elegant idea of capacity constraints, by which firms cannot sell more than they are able to produce. Since then, a vast economic literature on those Bertrand-Edgeworth models has been applied to a wide range of economic issues such as industrial organisation, macroecomics and international trade (see, e.g., Sogard 1996; Staiger and Wolak 1992; Iwand and Rosembaum 1991; Bjorsten 1994; Deneckere and Kovenock 1992).

However, most of those Bertrand-Edgeworth models failed to prove the general existence of a pure strategy equilibrium. They thus turn to the mixed strategy solution to avoid the non-existence problem. Nevertheless, mixed strategies are not uniformly accepted as a satisfactory explanation of pricing behavior by oligopoly firms (see, e.g. Friedman 1988; Dixon 1987; Levitan and Shubik 1980), although, in a large market and under some conditions, the mixed strategy outcome is not bothersome (see, Borgers 1992; Dixon 1987; Allen and Hellwig 1986a&b; Vives 1986). Of course, in a small industry for which the mixed strategy equilibrium does not tend to the competitive equilibrium at all, very interesting results have been found with models assuming sequential timing of firms moves (see, Shubik and Levitan 1980; Deneckere and Kovenock 1992; Canoy 1996). More recently, Díaz and Kujal (2002) introduces some grains of sand into those well-known models by imposing ex-ante the roles of Stackelberg leader-follower and by providing an alternative to the sequential timing hypothesis[1]. They show for a general class of rationing rules there exists a sub-game perfect equilibrium involving both firms playing pure strategies. Still, all those models did not succeed to go beyond the idea of *capacity constraints* elegantly introduced by Edgeworth (1897) more than a century ago.

The present paper, by contrast, shows that firms may be able *to set prices above the marginal costs and thus get positive profits.* This remarkable result is *robust to the number of firms and to cost asymmetries.*

[1] The term "grains of sand" is borrowed from Benabou-Tirole (2001).

Furthermore and more importantly, when firms' costs are different, *all firms* are awarded *positive profits* even though *they set price at the highest marginal cost.*

Contrary to the present literature, it gets rid of the common problem of capacity constraints. It neither considers any list pricing stage nor any subsequent price discounting stage nor any sequential timing of firms moves. It simply applies an innovative strategy where firms compete in a oligopoly market using a two-stage game. The key idea of this new way of competing is that each firm decides *unilaterally* to give away *voluntarily* a part of its profit to its rival[2]. Hence, each firm first (in the first-stage) chooses simultaneously the optimal part of its profit to give up to its rival and then (in the second-stage) determines consequently the equilibrium price.

The article proceeds as follows. Section 2 presents the model where firms have equal marginal costs. It first centers on the second-stage of the game and shows that there exists a multiplicity of NE_a. It then turns to the first-stage of the game and demonstrates the existence of a multiplicity of SPNE_a. It finally points out that firms may set prices above the marginal costs. Section 3 modifies the model by allowing firms to have different marginal costs. As in the previous section, solving first the second-stage and, then the first-stage, it highlights that firms' profits are also positive even though they set prices at the highest marginal cost. Section 4 and section 5 generalise the previous models to n firms respectively with equal and different marginal costs and thus shed light that our remarkable result is robust to the number of firms and to cost asymmetries. Section 6 concludes with suggestions for future research.

2 The model

We first consider two firms 1 and 2 in a homogeneous market[3]. We suppose that each firm incurs a cost c per unit of production[4]. The market demand function is $q = D(p) = 1 - p$. We assume that firms do not have capacity constraints and always supply the demand they face. Therefore, the profit function of firm i is:

[2] The rationale behind the "unilateral-decision" assumption is to support the legality of this strategy. Consequently, our firms should not be treated as a cartel or as colluding firms or as joint ventures.

[3] In section 4 and section 5, we will relax this assumption by generalising the model to n firms.

[4] In section 3 and section 5, we will relax this assumption by allowing firms to have different marginal costs.

$$\Pi_i = \begin{cases} (p_i - c)q_i & if \quad p_i < p_j \\ \frac{1}{2}(p_i - c)q_i & if \quad p_i = p_j \\ 0 & \quad otherwise \end{cases} \qquad i = 1, 2 \ (i \neq j)$$

where q_i is the quantity demanded faced by firm i.

Now, let us introduce a grain of novelty in the basic Bertrand model. Let α_1 (resp. α_2) denote the part of the profit that firm 1 (resp. firm 2) wants to share with firm 2 (resp. firm 1). We suppose that $\alpha_i \in]0, 1[$. Consequently, we can write the *new profit function* $P_i(p_i(\alpha_i, \alpha_j), p_j(\alpha_i, \alpha_j))$ (hereafter P_i) of each firm as:

$$P_i = (1 - \alpha_i)\Pi_i(p_i(\alpha_i, \alpha_j), p_j(\alpha_i, \alpha_j)) + \alpha_j \Pi_j(p_i(\alpha_i, \alpha_j), p_j(\alpha_i, \alpha_j))$$

We consider a two-stage game whose sequences are thus defined. In the first stage of the game, firms chooses α_i. In the second stage of the game, firms select p_i.

In the *first stage of the game*, for α_1 and α_2 firms simultaneously solve:

$$Max_{\alpha_1} \quad P_1 = (1 - \alpha_1)\Pi_1 + \alpha_2 \Pi_2$$

$$Max_{\alpha_2} \quad P_2 = (1 - \alpha_2)\Pi_2 + \alpha_1 \Pi_1$$

In the *second stage of game*, for p_1 and p_2 firms simultaneously solve:

$$Max_{p_1} \quad P_1 = (1 - \alpha_1)\Pi_1 + \alpha_2 \Pi_2$$

$$Max_{p_2} \quad P_2 = (1 - \alpha_2)\Pi_2 + \alpha_1 \Pi_1$$

2.1 Solving the second-stage of the game

To find the subgame perfect Nash equilibrium (SPNE), we begin by solving subgames in the second-stage. Recall that, in the second stage, firms are looking for prices that maximize their profits.

Proposition 1 *If $\alpha_1 + \alpha_2 = 1$, then any prices (p_1, p_2) such that $c \leq p_1 = p_2 \leq p_m$ are NE_a in the second stage of the game*

Proof. (p_1, p_2) such that $c \leq p_1 = p_2 \leq p_m$ are NE$_a$ if and only if no firm wants to deviate from those prices by fixing a price p_i' above or below. In fact:

$c \leq p_1 = p_2 = p \leq p_m \Rightarrow \Pi_1 = \Pi_2 \geq 0$

$\Pi_1 = \frac{1}{2}(p_1 - c)(1 - p_1) = \frac{1}{2}(p - c)(1 - p)$

$\Pi_2 = \frac{1}{2}(p_2 - c)(1 - p_2) = \frac{1}{2}(p - c)(1 - p)$

$P_1 = (1 - \alpha_1)\Pi_1 + \alpha_2\Pi_2 = (1 - \alpha_1)\frac{1}{2}(p - c)(1 - p) + \alpha_2\frac{1}{2}(p - c)(1 - p)$

$P_1 = \frac{1}{2}(1 - \alpha_1 + \alpha_2)(p - c)(1 - p)$

$P_2 = \frac{1}{2}(1 - \alpha_2 + \alpha_1)(p - c)(1 - p)$

Suppose that:

i) $p_1 = p_2 - \varepsilon$ $(c < p_1 < p_2) \Longleftrightarrow \Pi_1 = (1 - p_1)(p_1 - c) > 0$ and $\Pi_2 = 0$

$P_1' = (1 - \alpha_1)\Pi_1 = (1 - \alpha_1)(1 - p_1)(p_1 - c)$

If $p_1 \leq p_m$(monopolistic price), then $p_1 = p - \varepsilon$.

For ε very small[5], $P_1' \simeq (1 - \alpha_1)(1 - p)(p - c) \leq P_1 \Leftrightarrow$

$$(1 - \alpha_1) \leq \frac{1}{2}(1 - \alpha_1 + \alpha_2) \text{ or } \alpha_1 + \alpha_2 \geq 1 \tag{1}$$

ii) $p_1 = p_2 + \varepsilon$ $(p_1 > p_2 > c) \Longleftrightarrow \Pi_2 = (1 - p_2)(p_2 - c) > 0$ and $\Pi_1 = 0$

$P_1'' = \alpha_2\Pi_2 = \alpha_2(1 - p_2)(p_2 - c) = \alpha_2(1 - p)(p - c) \leq P_1 \Leftrightarrow$

$$\alpha_2 \leq \frac{1}{2}(1 - \alpha_1 + \alpha_2) \text{ or } \alpha_1 + \alpha_2 \leq 1 \tag{2}$$

Equations (1) and (2) represent the non-deviation conditions and are both satisfied when $\alpha_1 + \alpha_2 = 1$

Conclusion: if $\alpha_1 + \alpha_2 = 1$, (p_1, p_2) such that $c \leq p_1 = p_2 \leq p_m$ are NE$_a$ in the second-stage of the game. ■

[5] There is no reason for not to suppose that ε is very small. For instance, firms need to decrease or increase just slightly to get or to lose the entire market.

Proposition 2 *If $\alpha_1 + \alpha_2 > 1$, then any prices (p_i, p_j) such that $c \leq p_i = p_m < p_j$ are NE$_a$ in the second stage of the game*

Proof. (p_1, p_2) s. t. $c \leq p_2 = p_m < p_1$ are NE$_a$ if and only if no firm has interest to deviate from those prices by fixing a price p_i' above or below.

$c \leq p_2 = p_m < p_1 \Rightarrow \Pi_1 = 0$ and $\Pi_2 = (p_2 - c)(1 - p_2) > 0$

$P_1 = \alpha_2 \Pi_2 = \alpha_2 (p_2 - c)(1 - p_2)$

$P_2 = (1 - \alpha_2) \Pi_2 = (1 - \alpha_2)(p_2 - c)(1 - p_2)$

Since prices p_1 and p_2 are different, we have to study separately the deviation for both firms. Let us check first for firm 1. Suppose that:

i) $p_1 = p_2 - \varepsilon$ $(c < p_1 < p_2) \Longleftrightarrow \Pi_1 = (1 - p_1)(p_1 - c)$ and $\Pi_2 = 0$

$P_1' = (1 - \alpha_1) \Pi_1 = (1 - \alpha_1)(1 - p_2 + \varepsilon)(p_2 - \varepsilon - c)$

For ε very small, $P_1' \simeq (1 - \alpha_1)(1 - p_2)(p_2 - c) < P_1 \Leftrightarrow$

$$(1 - \alpha_1) < \alpha_2 \text{ or } \alpha_1 + \alpha_2 > 1 \tag{3}$$

ii) $p_1 = p_2 + \varepsilon$ $(p_1 > p_2 > c) \Longleftrightarrow \Pi_2 = (1 - p_2)(p_2 - c) > 0$ and $\Pi_1 = 0$

$$P_1' = \alpha_2 \Pi_2 = \alpha_2 (1 - p_2)(p_2 - c) = P_1, \forall \alpha_2 \tag{4}$$

Equations (3) and (4) represent the non-deviation conditions for firm 1 and are both satisfied when $\alpha_1 + \alpha_2 > 1$

Now, let us check for firm 2. Suppose that[6]:

i) $p_2' = p_2 + \varepsilon$ $(p_2' = p_m \;\&\; p_2' < p_1) \Leftrightarrow \Pi_1 = 0$ and $\Pi_2 > 0$

$P_2' = (1 - \alpha_2)(1 - p_2 - \varepsilon)(p_2 - \varepsilon - c)$

For ε very small, $P_2' \simeq P_2$ and firm 2 has no interest to deviate

Conclusion: if $\alpha_1 + \alpha_2 > 1$, (p_i, p_j) such that $c \leq p_i = p_m < p_j$ are NE$_a$ in the second-stage of the game. ■

[6] We do not need to suppose that $p_2' = p_2 - \varepsilon$. Firm 2, being alone and therefore controlling the entire market, has no interest to decrease its price even slightly. However, it could always try to increase its price just a little bit to get more profit.

Proposition 3 *If $\alpha_1+\alpha_2<1$, then any prices $(p_1,\, p_2)$ such that $p_1=p_2=c$ is NE in the second stage of the game*

Proof. $(p_1,\, p_2)$ s.t. $p_1=p_2=c$ is NE if and only if no firm has interest to deviate from those prices to fix a price p_i' above or below. Furthermore, there does not exist any other equilibrium prices. First, let us show that $(p_1,\, p_2)$ s.t. $p_1=p_2=c$ is a NE.

$p_2=p_2=c \Rightarrow \Pi_1=0$ and $\Pi_2=0$

$P_1=(1-\alpha_1)\,\Pi_1+\alpha_2\Pi_2=0$

$P_2=(1-\alpha_2)\,\Pi_2+\alpha_1\Pi_1=0$

Suppose that:

i) $p_1=p_2-\varepsilon$ ($p_1<p_2$ and $p_1<c$) $\Rightarrow$ $\Pi_1=(1-p_1)\,(p_1-c)<0$ and $\Pi_2=0$

$P_1'=(1-\alpha_1)\,\Pi_1=(1-\alpha_1)\,(1-p_1)\,(p_1-c)<0$

$P_1'=(1-\alpha_1)\,(1-p_1)\,(p_1-c)<P_1=0$ $\Rightarrow$Firm 1 has no interest by fixing a price below p_2

ii) $p_1=p_2+\varepsilon$ ($p_1>p_2=c$) $\Longleftrightarrow$ $\Pi_2=(1-p_2)\,(p_2-c)=0$ and $\Pi_1=0$ (firm 1 does not produce)

$P_1''=\alpha_2\Pi_2=\alpha_2\,(1-p_2)\,(p_2-c)=P_1=0$ $\Rightarrow$Firm 1 has no interest by fixing a price above p_2

Conclusion: if $\alpha_1+\alpha_2<1$, $(p_1,\, p_2)$ s.t. $p_1=p_2=c$ constitute a NE in the second-stage of the game.

Now, let us show that there does not exist other prices equilibria for $\alpha_1+\alpha_2<1$. Let us consider different other prices scenarios. Suppose that: $c<p_1=p_2=p<p_m$

$c<p_1=p_2=p<p_m \Rightarrow \Pi_1=\frac{1}{2}\,(p-c)\,(1-p)=\Pi_2$

$P_1=(1-\alpha_1)\,\Pi_1+\alpha_2\Pi_2=(1-\alpha_1)\,\frac{1}{2}\,(p-c)\,(1-p)+\alpha_2\frac{1}{2}\,(p-c)\,(1-p)$ or

$P_1=\frac{1}{2}\,(1-\alpha_1+\alpha_2)\,(1-p)\,(p-c)$

We know that: $\alpha_1 + \alpha_2 < 1 \Rightarrow 1 - \alpha_1 + \alpha_2 > 2\alpha_2 \Rightarrow \exists\ R > 0$: $1 - \alpha_1 + \alpha_2 = 2\alpha_2 + R$ or $1 - \alpha_1 = \alpha_2 + R$ or $R = 1 - \alpha_1 - \alpha_2$ or $\alpha_2 = 1 - \alpha_1 - R$

$\Rightarrow P_1 = \left(\alpha_2 + \frac{1}{2}R\right)(1-p)(p-c)$

Can (p_1, p_2) s.t. $c < p_1 = p_2 = p < p_m$ be NE? Suppose that:

i) $p_1 = p_2 - \varepsilon$ $(c < p_1 < p_2) \Rightarrow \Pi_1 = (1-p_1)(p_1-c) > 0$ and $\Pi_2 = 0$

$P_1' = (1-\alpha_1)\Pi_1 = (1-\alpha_1)(1-p_1)(p_1-c)$ or

$P_1' = (1-\alpha_1)(1-p_2+\varepsilon)(p_2-c-\varepsilon)$

For ε very small, $P_1' \simeq (1-\alpha_1)(1-p)(p-c) = (\alpha_2+R)(1-p)(p-c)$

$\Rightarrow P_1' > P_1,\ \forall\ R > 0$. Thus, firm 1 has interest to deviate from $c < p_1 = p_2 = p < p_m$

Consequently, if $\alpha_1 + \alpha_2 < 1$, (p_1, p_2) s.t. $c < p_1 = p_2 = p < p_m$ is not a NE in the second-stage of the game.

Using the same reasoning as before, we can show that if $\alpha_1 + \alpha_2 < 1$, then (p_1, p_2) s.t. $c < p_1 = p_2 = p > p_m$ is not a NE in the second-stage of the game.

Likewise, it is easy to show that if $\alpha_1 + \alpha_2 < 1$, then any other prices scenarios different from $p_1 = p_2 = c$ are not NE_a. One can check that any prices (p_1, p_2) such that $c < p_1 < p_2 < p_m$ or $c < p_1 < p_2 = p_m$ or $c < p_1 < p_2 > p_m$ are not NE_a since one firm has always interest to deviate. For example, suppose that: $c < p_1 < p_2 < p_m$

$c < p_2 < p_1 < p_m \Rightarrow \Pi_2 = (p_2-c)(1-p_2) > 0$ and $\Pi_1 = 0$

$P_1 = \alpha_2\Pi_2 = \alpha_2(1-p_2)(p_2-c)$

$P_2 = (1-\alpha_2)\Pi_2 = (1-\alpha_2)(1-p_2)(p_2-c)$

Can (p_1, p_2) s.t. $c < p_2 < p_1 < p_m$ be NE? Suppose that:

i) $p_1 = p_2 - \varepsilon$ $(c < p_1 < p_2) \Rightarrow \Pi_1 = (1-p_1)(p_1-c) > 0$ and $\Pi_1 = 0$

$P_1' = (1-\alpha_1)\Pi_1 = (1-\alpha_1)(1-p_1)(p_1-c)$

$P_1' = (1-\alpha_1)(1-p_2+\varepsilon)(p_2-c-\varepsilon)$

For ε very small, $P_1' \simeq (1-\alpha_1)(1-p_2)(p_2-c) = (\alpha_2 + R)(1-p_2)(p_2-c)$

$\Rightarrow P_1' > P_1$ since $R > 0$. Firm 1 thus has interest to deviate from $c < p_1 < p_2 < p_m$

Consequently, if $\alpha_1 + \alpha_2 < 1$, (p_1, p_2) s.t. $c < p_2 < p_1 < p_m$ is not a NE in the second-stage of the game.

Using the same reasoning as before, we can show that if $\alpha_1 + \alpha_2 < 1$, then (p_1, p_2) s.t. $c < p_1 < p_2 > p_m$ is not a NE in the second-stage of the game.

Conclusion: if $\alpha_1 + \alpha_2 < 1$, (p_1, p_2) s.t.$p_1 = p_2 = c$ is a NE in the second-stage of the game. ■

The second-stage being entirely solved and NE$_a$ being found, we can thus move to the first-stage of the game in order to find SPNE$_a$

2.2 Solving the first-stage of the game

In the first-stage of the game, firms choose the α_i optimal maximizing their profit to share with their rival.

Solving backwards, we have solved the second-stage of the game in the previous section and have found NE$_a$ in prices summarized below:

i) $(p_1, p_2) : p_1 = p_2 = c$ if $\alpha_1 + \alpha_2 < 1$ with:

$$\begin{cases} P_1 = 0 \\ P_2 = 0 \end{cases}$$

ii) $(p_1, p_2) : c \leq p_1 = p_2 = p \leq p_m$ if $\alpha_1 + \alpha_2 = 1$ with:

$$\begin{cases} P_1 = \frac{1}{2}(1-\alpha_1+\alpha_2)(p-c)(1-p) = \alpha_2 (p-c)(1-p) \\ P_2 = \frac{1}{2}(1-\alpha_2+\alpha_1)(p-c)(1-p) = \alpha_1 (p-c)(1-p) \end{cases}$$

iii) $(p_1, p_2) : c \leq p_i = p_m < p_j$ if $\alpha_1 + \alpha_2 > 1$ with:

$$\begin{cases} P_1 = \alpha_2 (p_m - c)(1-p_m) \\ P_2 = (1-\alpha_2)(p_m - c)(1-p_m) \end{cases}$$

Now, in the current section, we draw our attention to the first-stage of the game searching for SPNE$_a$ in α_i.

Proposition 4 *The strategies* $(\alpha_1, p_1(\alpha_1, \alpha_2))$, $(\alpha_1, p_1(\alpha_1, \alpha_2))$ *s.t.:*

i) $\alpha_i \in]0,1[\ \ \&\ \ \alpha_1 + \alpha_2 = 1$

$$ii) \begin{cases} p_1^* = p_2^* = c \ if \ \ \alpha_1 + \alpha_2 < 1 \\ p_1^* = p_2^* = p_m \ if \ \ \alpha_1 + \alpha_2 = 1 \\ c \leq p_i = p_m < p_j \ if \ \ \alpha_1 + \alpha_2 > 1 \end{cases}$$

are SPNE$_a$ *of the game. Furthermore, if* $\alpha_j > 0$, *then firm i's profits in the SNPE*$_a$ *are* $\alpha_j (p_m - c)(1 - p_m)$ *higher than in the case where* $\alpha_1 = \alpha_2 = 0$.

Proof. Let us show the first part of the proposition.

The strategies $(\alpha_1, p_1(\alpha_1, \alpha_2))$, $(\alpha_1, p_1(\alpha_1, \alpha_2))$ s.t. *i)* and *ii)* are satisfied, are SPNE$_a$ if and only if no firm has interest to deviate from those prices by choosing a α_i' above or below. Furthermore, there does not exist any other SPNE in first-stage of the game. Because of the multiplicity of α_i, we investigate separately the deviation for each firm.

Let us check first for firm 1. Suppose that:

i) $\alpha_1' < \alpha_1 \Rightarrow \alpha_1' + \alpha_2 < 1 \ (\alpha_1 + \alpha_2 = 1) \Rightarrow$

$$P_1' = 0 < P_1 = \alpha_2 (p_m - c)(1 - p_m) \tag{5}$$

ii) $\alpha_1' > \alpha_1 \Rightarrow \alpha_1' + \alpha_2 > 1 \ (\alpha_1 + \alpha_2 = 1) \Rightarrow$

$$P_1'' = \alpha_2 (p_m - c)(1 - p_m) = P_1 = \alpha_2 (p_m - c)(1 - p_m) \tag{6}$$

(5) and (6) show that firm 1 has no interest to deviate.

Now, let us check for firm 2. Suppose that:

i) $\alpha_2' < \alpha_2 \Rightarrow \alpha_1' + \alpha_2 < 1 \ (\alpha_1 + \alpha_2 = 1) \Rightarrow$

$$P_2' = 0 < P_2 = (1 - \alpha_2)(p_m - c)(1 - p_m) \tag{7}$$

ii) $\alpha_2' > \alpha_2 \Rightarrow \alpha_2' + \alpha_1 > 1 \ (\alpha_1 + \alpha_2 = 1) \Rightarrow$

$$P_2'' = (1 - \alpha_2')(p_m - c)(1 - p_m) < P_2 = \alpha_1 (p_m - c)(1 - p_m) \tag{8}$$

(7) and (8) show that firm 2 has no interest to deviate.

Conclusion: The strategies $(\alpha_1, p_1(\alpha_1, \alpha_2))$, $(\alpha_1, p_1(\alpha_1, \alpha_2))$ s.t. *i)* and *ii)* are satistified, are SPNE$_a$

Now, the question that remains is whether there exists other NE_a different from those above. A good candidate would be the strategies $(\alpha_1, p_1(\alpha_1, \alpha_2))$, $(\alpha_1, p_1(\alpha_1, \alpha_2))$ s.t.:

i) $\alpha_i \in \,]0, 1[\;\; \& \;\; \alpha_1 + \alpha_2 = 1$

$$ii) \begin{cases} p_1^* = p_2^* = c \; if \;\; \alpha_1 + \alpha_2 < 1 \\ p_1^* = p_2^* \leq p_m \; if \;\; \alpha_1 + \alpha_2 = 1 \\ c \leq p_i = p_m < p_j \; if \;\; \alpha_1 + \alpha_2 > 1 \end{cases}$$

since we have found that, if $\alpha_1 + \alpha_2 = 1$ then (p_1, p_2) s.t. $c \leq p_1^* = p_2^* \leq p_m$ were NE_a in the second-stage of the game. Note that $\alpha_1 + \alpha_2 = 1$ with $c \leq p_1^* = p_2^* = p \leq p_m \Rightarrow P_1 = \alpha_2 (p - c)(1 - p)$ and $P_2 = \alpha_1 (p - c)(1 - p)$.

Let us show that $(\alpha_1, \alpha_2) : \alpha_i \in \,]0, 1[\;\; \& \;\; \alpha_1 + \alpha_2 = 1$; $(p_1, p_2) : p_1^* = p_2^* = c$ if $\alpha_1 + \alpha_2 < 1$ & $p_1^* = p_2^* \leq p_m$ if $\alpha_1 + \alpha_2 = 1$ & $c \leq p_i = p_m < p_j$ if $\alpha_1 + \alpha_2 > 1$ could not be SPNE_a. For that, it suffices to prove that one firm has interest to deviate. Suppose that:

i) $\alpha_1' < \alpha_1 \Rightarrow \alpha_1' + \alpha_2 < 1 \; (\alpha_1 + \alpha_2 = 1) \Rightarrow$

$P_1' = 0 < P_1 = \alpha_2 (p - c)(1 - p)$

ii) $\alpha_1' > \alpha_1 \Rightarrow \alpha_1' + \alpha_2 > 1 \; (\alpha_1 + \alpha_2 = 1) \Rightarrow$

$$P_1'' = \alpha_2 (p_m - c)(1 - p_m) > P_1 = \alpha_2 (p - c)(1 - p) \tag{9}$$

Equation (9) says that firm 1 would deviate and therefore, $(\alpha_1, \alpha_2) : \alpha_i \in \,]0, 1[\;\; \& \;\; \alpha_1 + \alpha_2 = 1$; $(p_1, p_2) : p_1^* = p_2^* = c$ if $\alpha_1 + \alpha_2 < 1$ & $p_1^* = p_2^* \leq p_m$ if $\alpha_1 + \alpha_2 = 1$ & $c \leq p_i = p_m < p_j$ if $\alpha_1 + \alpha_2 > 1$ cannot be SPNE_a

Likewise, we can show that any other pair $(\alpha_1, \alpha_2) : \alpha_1 + \alpha_2 \neq 1$ cannot be SPNE_a. The intuition behind is simple. No firm is interested in the case where $\alpha_1 + \alpha_2 < 1$ since it leads to $P_1 = P_2 = 0$ as we have seen before in the beginning of this section.

Now, the last case that remains, is when $\alpha_1 + \alpha_2 > 1$. Recall that in this case, we have found prices equilibria $(p_1, p_2) : c \leq p_i = p_m < p_j$ with $P_1 = \alpha_2 (p_m - c)(1 - p_m)$ and $P_2 = (1 - \alpha_2)(p_m - c)(1 - p_m)$. Therefore, this situation is tempting for firm 1 since it would get $P_1 = \alpha_2 (p_m - c)(1 - p_m)$ *even though it gives nothing.* However, this case is detrimental to firm 2 since it is left with $P_2 = (1 - \alpha_2)(p_m - c)(1 - p_m)$. Thus, firm 2 would like α_2 as small as possible. However, it cannot decrease α_2 too much for fear that $\alpha_1 + \alpha_2 < 1$. Otherwise, it would get zero profits ($P_2 = 0$). Its only favorable situation is when $\alpha_1 + \alpha_2 = 1$. So, any pair (α_1, α_2) such that $\alpha_1 + \alpha_2 > 1$ cannot be NE_a.

Finally, we conclude that The strategies $(\alpha_1, p_1(\alpha_1, \alpha_2))$, $(\alpha_1, p_1(\alpha_1, \alpha_2))$ s.t. $i)$ and $ii)$ are satisfied, are SPNE$_a$

The second part of the proposition is straightforward. We all know the common result of the Bertrand paradox where both prices $\left(p_i^b\right)$ are equal to marginal costs and profits $\left(P_i^b\right)$ are zero[7]. Hence, the difference between the both profits is:

$$P_i - P_i^b = \alpha_j (p_m - c)(1 - p_m) - 0 = \alpha_j (p_m - c)(1 - p_m)$$

Conclusion: If $\alpha_j > 0$, then firm i's profits in the SPNE$_a$ are $\alpha_j (p_m - c)(1 - p_m)$ higher than in the case where $\alpha_1 = \alpha_2 = 0$. ■

3 The modified model

We consider the same model as before except that we allow firms to have different marginal costs. We still consider two firms 1 and 2 in a homogeneous market. Now, we suppose that each firm incurs a cost c_i $(c_1 < c_2)$ per unit of production. Therefore, the profit function of firm i becomes:

$$\Pi_i = \begin{cases} (p - c_i)q_i & if \quad p_i < p_j \\ \frac{1}{2}(p - c_i)q_i & if \quad p_i = p_j \\ 0 & \quad\quad otherwise \end{cases} \qquad i = 1, 2\ (i \neq j)$$

where q_i is the quantity demanded faced by firm i.

Now, let us introduce a grain of novelty in the basic Bertrand model. Let α_1 (resp. α_2) denote the part of the profit that firm 1 (resp. firm 2) wants to share with firm 2 (resp. firm 1). We suppose that $\alpha_i \in]0, 1[$. Consequently, we can write the *new profit function* $P_i(p_i(\alpha_i, \alpha_j), p_j(\alpha_i, \alpha_j))$ (hereafter P_i) of each firm as:

$$P_i = (1 - \alpha_i)\Pi_i(p_i(\alpha_i, \alpha_j), p_j(\alpha_i, \alpha_j) + \alpha_j \Pi_j(p_i(\alpha_i, \alpha_j), p_j(\alpha_i, \alpha_j))$$

We consider a two-stage game whose sequences are thus defined. In the first stage of the game, firms chooses α_i. In the second stage of the game, firms select p_i.

In the *first stage of the game*, for α_1 and α_2 firms simultaneously solve:

[7] To avoid confusion with our model, we denote by p_i^b (resp. P_i^b) the prices (resp. the profits) in the basic Betrand model.

$$Max_{a_1} \quad P_1 = (1-\alpha_1)\Pi_1 + \alpha_2\Pi_2$$

$$Max_{\alpha_2} \quad P_2 = (1-\alpha_2)\Pi_2 + \alpha_1\Pi_1$$

In the *second stage of game*, for p_1 and p_2 firms simultaneously solve:

$$Max_{p_1} \quad P_1 = (1-\alpha_1)\Pi_1 + \alpha_2\Pi_2$$

$$Max_{p_2} \quad P_2 = (1-\alpha_2)\Pi_2 + \alpha_1\Pi_1$$

3.1 Solving the second-stage of the game

To find the subgame perfect Nash equilibrium (SPNE), we begin by solving subgames in the second-stage. Recall that, in the second stage, firms are looking for prices that maximize their profits.

Proposition 5 *If $\alpha_1 + \alpha_2 = 1$, then any prices (p_1, p_2) such that $c_2 \leq p_1 = p_2 \leq p_m^2$ (firm 2's monopolistic price) are NE_a in the second stage of the game*

Proof. (p_1, p_2) such that $c_2 \leq p_1 = p_2 \leq p_m$ are NE_a if and only if no firm wants to deviate from those prices by fixing a price p_i' above or below. In fact:

$c_2 \leq p_1 = p_2 = p \leq p_m \Rightarrow \Pi_1, \Pi_2 \geq 0$

$\Pi_1 = \frac{1}{2}(p_1 - c_1)(1-p_1) = \frac{1}{2}(p-c_1)(1-p)$

$\Pi_2 = \frac{1}{2}(p_2 - c_2)(1-p_2) = \frac{1}{2}(p-c_2)(1-p)$

$P_1 = (1-\alpha_1)\Pi_1 + \alpha_2\Pi_2 = (1-\alpha_1)\frac{1}{2}(p-c_1)(1-p) + \alpha_2\frac{1}{2}(p-c_2)(1-p)$

$P_1 = \frac{1}{2}(1-p)\left[(1-\alpha_1)(p-c_1) + \alpha_2(p-c_2)\right]$

$P_2 = \frac{1}{2}(1-p)\left[(1-\alpha_2)(p-c_2) + \alpha_1(p-c_1)\right]$

Since prices p_1 and p_2 are different, we have to study separately the deviation for both firms. Let us check first for firm 1. Suppose that:

i) $p_1 = p_2 - \varepsilon \ (\varepsilon > 0) \Longleftrightarrow \Pi_1 = (1-p_1)(p_1 - c_1) > 0$ and $\Pi_2 = 0$

$P_1' = (1-\alpha_1)\,\Pi_1 = (1-\alpha_1)\,(1-p+\varepsilon)\,(p-c_1-\varepsilon)$

For ε very small[8], $P_1' \simeq (1-\alpha_1)\,(1-p)\,(p-c_1) \leq P_1 \Leftrightarrow$

$(1-\alpha_1)\,(p-c_1) \leq \frac{1}{2}\left[(1-\alpha_1)\,(p-c_1)+\alpha_2\,(p-c_2)\right]$ or

$$\frac{1-\alpha_1}{\alpha_2} \leq \frac{p-c_2}{p-c_1} \tag{10}$$

ii) $p_1 = p_2+\varepsilon$ $(\varepsilon>0) \Longleftrightarrow \Pi_2 = (1-p_2)\,(p_2-c_2) > 0$ and $\Pi_1 = 0$

$P_1'' = \alpha_2\Pi_2 = \alpha_2\,(1-p_2)\,(p_2-c_2) = \alpha_2\,(1-p)\,(p-c_2) \leq P_1 \Leftrightarrow$

$\alpha_2\,(p-c_2) \leq \frac{1}{2}\left[(1-\alpha_1)\,(p-c_1)+\alpha_2\,(p-c_2)\right]$ or

$$\frac{p-c_2}{p-c_1} \leq \frac{1-\alpha_1}{\alpha_2} \tag{11}$$

Equations (10) and (11) represent the non-deviation conditions for firm 1 and are both satisfied when $\frac{p-c_2}{p-c_1} = \frac{1-\alpha_1}{\alpha_2}$

Now, let us check for firm 2. Suppose that:

i) $p_2 = p_1-\varepsilon$ $(\varepsilon>0) \Longleftrightarrow \Pi_2 = (1-p_2)\,(p_2-c_2) > 0$ and $\Pi_1 = 0$

$P_2' = (1-\alpha_2)\,\Pi_2 = (1-\alpha_2)\,(1-p+\varepsilon)\,(p-c_2-\varepsilon)$

For ε very small[9], $P_2' \simeq (1-\alpha_2)\,(1-p)\,(p-c_2) \leq P_2 \Leftrightarrow$

$(1-\alpha_2)\,(p-c_2) \leq \frac{1}{2}\left[(1-\alpha_2)\,(p-c_2)+\alpha_1\,(p-c_1)\right]$ or

$$\frac{p-c_2}{p-c_1} \leq \frac{\alpha_1}{1-\alpha_2} \tag{12}$$

ii) $p_2 = p_1+\varepsilon$ $(\varepsilon>0) \Longleftrightarrow \Pi_1 = (1-p_1)\,(p_1-c_1) > 0$ and $\Pi_2 = 0$

$P_2'' = \alpha_1\Pi_1 = \alpha_1\,(1-p_1)\,(p_1-c_1) = \alpha_1\,(1-p)\,(p-c_1) \leq P_2 \Leftrightarrow$

$\alpha_1\,(p-c_1) \leq \frac{1}{2}\left[(1-\alpha_2)\,(p-c_2)+\alpha_1\,(p-c_1)\right]$ or

$$\frac{\alpha_1}{1-\alpha_2} \leq \frac{p-c_2}{p-c_1} \tag{13}$$

[8] There is no reason for not to suppose that ε is very small. For instance, firms need to decrease or increase just slightly to get or to lose the entire market.

[9] There is no reason for not to suppose that ε is very small. For instance, firms need to decrease or increase just slightly to get or to lose the entire market.

Equations (12) and (13) represent the non-deviation conditions for firm 2 and are both satisfied when $\frac{p-c_2}{p-c_1} = \frac{\alpha_1}{1-\alpha_2}$

Equations $(10)-(13)$ represent the non-deviation conditions for both firm and are both satisfied when $\frac{1-\alpha_1}{\alpha_2} = \frac{\alpha_1}{1-\alpha_2}$, that is, $\alpha_1 + \alpha_2 = 1$

Conclusion: if $\alpha_1 + \alpha_2 = 1$, (p_1, p_2) such that $c_2 \leq p_1 = p_2 \leq p_m$ are NE_a in the second-stage of the game. ■

Proposition 6 *If $\alpha_1 + \alpha_2 > 1$, then any prices (p_i, p_j) such that $c_2 \leq p_i = p_m^2 < p_j$ are NE_a in the second stage of the game.*

Proof. (p_1, p_2) s. t. $c_2 \leq p_2 = p_m^2 < p_1$ are NE_a if and only if no firm has interest to deviate from those prices by fixing a price p_i' above or below.

$c \leq p_2 = p_m^2 < p_1 \Rightarrow \Pi_1 = 0$ and $\Pi_2 = (p_2 - c_2)(1 - p_2) > 0$

$P_1 = \alpha_2 \Pi_2 = \alpha_2 (p_2 - c_2)(1 - p_2)$

$P_2 = (1 - \alpha_2)\Pi_2 = (1 - \alpha_2)(p_2 - c_2)(1 - p_2)$

Since prices p_1 and p_2 are different, we have to study separately the deviation for both firms. Let us check first for firm 1. Suppose that:

i) $p_1 = p_2 - \varepsilon$ $(\varepsilon > 0) \Longleftrightarrow \Pi_1 = (1 - p_1)(p_1 - c_1)$ and $\Pi_2 = 0$

$P_1' = (1 - \alpha_1)\Pi_1 = (1 - \alpha_1)(1 - p_2 + \varepsilon)(p_2 - \varepsilon - c_1)$

For ε very small, $P_1' \simeq (1 - \alpha_1)(1 - p_2)(p_2 - c_1) < P_1 \Leftrightarrow$

$(1 - \alpha_1)(1 - p_2)(p_2 - c_1) < \alpha_2 (p_2 - c_2)(1 - p_2)$ or

$$\frac{1 - \alpha_1}{\alpha_2} < \frac{p_2 - c_2}{p_2 - c_1} \tag{14}$$

Let us check now for firm 2. Suppose that:

i) $p_2 = p_1 - \varepsilon$ $(\varepsilon > 0) \Longleftrightarrow \Pi_1 = (1 - p_1)(p_1 - c_1)$ and $\Pi_2 = 0$

$P_2' = \alpha_1 \Pi_1 = \alpha_1 (1 - p_2 + \varepsilon)(p_2 - \varepsilon - c_1)$

For ε very small, $P_2' \simeq \alpha_1 (1 - p_2)(p_2 - c_1) < P_2 \Leftrightarrow$

$\alpha_1 (1 - p_2)(p_2 - c_1) < (1 - \alpha_2)(p_2 - c_2)(1 - p_2)$ or

$$\frac{p_2 - c_2}{p_2 - c_1} < \frac{\alpha_1}{1 - \alpha_2} \tag{15}$$

Equations (14) and (15) represent the non-deviation conditions for firm 1 and firm 2 and are both satisfied when $\frac{1-\alpha_1}{\alpha_2} < \frac{p_2-c_2}{p_2-c_1} < \frac{\alpha_1}{1-\alpha_2}$ or $\frac{1-\alpha_1}{\alpha_2} < \frac{\alpha_1}{1-\alpha_2}$ or $\alpha_1 + \alpha_2 > 1$

Conclusion: if $\alpha_1 + \alpha_2 > 1$, (p_i, p_j) such that $c_2 \leq p_i = p_m^2 < p_j$ constitute a NE_a in the second-stage of the game. ■

Proposition 7 *If $\alpha_1 + \alpha_2 < 1$, then any prices (p_1, p_2) such that $p_1 = c_2 - \varepsilon$ $(\varepsilon > 0)$ and $p_2 = c_2$ are NE in the second stage of the game.*

Proof. (p_1, p_2) s.t. $p_1 = c_2 - \varepsilon$ $(\varepsilon > 0)$ and $p_2 = c_2$ are NE if and only if no firm has interest to deviate from those prices to fix a price p_i' above or below.

$p_1 = c_2 - \varepsilon$ and $p_2 = c_2 \Rightarrow \Pi_1 = (p_1 - c_1)(1 - p_1) > 0$ and $\Pi_2 = 0$

$P_1 = (1 - \alpha_1)\Pi_1 = (1 - \alpha_1)(p_1 - c_1)(1 - p_1)$

$P_2 = \alpha_1 \Pi_1 = \alpha_1 (p_1 - c_1)(1 - p_1)$

Since costs c_1 and c_2 are different, we have to study separately the deviation for both firms. Let us check first for firm 1. Suppose that:

i) $p_1' < p_1 \Rightarrow \Pi_1 = (1 - p_1')(p_1' - c_1) > 0$ and $\Pi_2 = 0$

$P_1' = (1 - \alpha_1)\Pi_1 = (1 - \alpha_1)(1 - p_1')(p_1' - c_1) > 0$

$P_1' = (1 - \alpha_1)(1 - p_1')(p_1' - c_1) \leq P_1 \Rightarrow$Firm 1 has no interest by fixing a price below p_2

ii) $p_1'' = c_2 > p_1 \Rightarrow \Pi_1 = \frac{1}{2}(1 - p_1'')(p_1'' - c_1) = 0$ and $\Pi_2 = 0$

$P_1'' < P_1 \Rightarrow$Firm 1 has no interest by fixing a price above p_1

Now, let us check for firm 2. Suppose that:

i) $p_2' < p_1 \Rightarrow \Pi_2 < 0$ and $\Pi_1 = 0$ (firm 1 out of the market)

$P_2' = (1-\alpha_2)\,\Pi_2 < 0 < P_2 \Rightarrow$Firm 1 has no interest by fixing a price below p_2

ii) $p_2'' > p_1 \Rightarrow \Pi_2 = 0$ and $\Pi_1 = (1-p_1)(p_1-c_1) > 0$

$P_2'' = \alpha_1\Pi_1 = \alpha_1(1-p_1)(p_1-c_1) = P_2 \Rightarrow$Firm 1 has no interest by fixing a price above p_2

Conclusion: if $\alpha_1+\alpha_2 < 1$, (p_1, p_2) s.t. $p_1 = c_2 - \varepsilon$ $(\varepsilon > 0)$ and $p_2 = c_2$ are NE in the second-stage of the game. ■

Note that, in the last NE *firms' profits are positive even though they set price at the highest marginal cost.*

The second-stage being entirely solved and NE_a being found, we can thus move to the first-stage of the game in order to find SPNE_a

3.2 Solving the first-stage of the game

In the first-stage of the game, firms choose the α_i optimal maximizing their profits to share with their rivals.

Solving backwards, we have solved the second-stage of the game in the previous section and have found NE_a in prices summarized below:

i) $(p_1, p_2) : p_1 = c_2 - \varepsilon$ $(\varepsilon > 0)$ and $p_2 = c_2$ if $\alpha_1+\alpha_2 < 1$ with:

$$\begin{cases} P_1 = (1-\alpha_1)(p_1-c_1)(1-p_1) \\ P_2 = \alpha_1(p_1-c_1)(1-p_1) \end{cases}$$

ii) $(p_1, p_2) : c_2 \leq p_1 = p_2 = p \leq p_m^2$ if $\alpha_1+\alpha_2 = 1$ with:

$$\begin{cases} P_1 = \frac{1}{2}(1-p)\left[(1-\alpha_1)(p-c_1) + \alpha_2(p-c_2)\right] \approx \alpha_2(p-c_2)(1-p) \\ P_2 = \frac{1}{2}(1-p)\left[(1-\alpha_2)(p-c_2) + \alpha_1(p-c_1)\right] \approx \alpha_1(p-c_2)(1-p) \end{cases}$$

iii) $(p_1, p_2) : c_2 \leq p_i = p_m^2 < p_j$ if $\alpha_1+\alpha_2 > 1$ with:

$$\begin{cases} P_1 = \alpha_2(p_m^2-c_2)(1-p_m^2) \\ P_2 = (1-\alpha_2)(p_m^2-c_2)(1-p_m^2) \end{cases}$$

Note that in every NE, firms are awarded positive profits. *This is the main difference with the previous model where firms have equal marginal costs.*

Now, in the current section, we draw our attention to the first-stage of the game searching for SPNE_a in α_i.

Proposition 8 *The strategies* $(\alpha_1, p_1(\alpha_1, \alpha_2))$, $(\alpha_1, p_1(\alpha_1, \alpha_2))$ *s.t.:*

i) $\alpha_i \in \,]0,1[\ \ \& \ \ \alpha_1 + \alpha_2 = 1$

ii) $\begin{cases} p_1 = c_2 - \varepsilon\,(\varepsilon > 0)\ \ \&\ p_2 = c_2\ if\ \alpha_1 + \alpha_2 < 1 \\ p_1^* = p_2^* = p_m^2\ if\ \alpha_1 + \alpha_2 = 1 \\ c_2 \leq p_i = p_m^2 < p_j\ if\ \alpha_1 + \alpha_2 > 1 \end{cases}$

are SPNE$_a$ *of the game. Furthermore, if* $\alpha_j > 0$*, then firm i's profits in the SNPE are* $\alpha_j\,(p_m^2 - c_2)\,(1 - p_m^2)$ *higher than in the case where* $\alpha_1 = \alpha_2 = 0$*.*

Proof. Let us show the first part of the proposition.

The strategies $(\alpha_1, p_1(\alpha_1, \alpha_2))$, $(\alpha_1, p_1(\alpha_1, \alpha_2))$ s.t. *i)* and *ii)* are satisfied, are SPNE$_a$ if and only if no firm has interest to deviate from those prices by choosing a α_i' above or below. Because of the multiplicity of α_i, we investigate separately the deviation for each firm.

Let us check first for firm 1. Suppose that:

i) $\alpha_1' < \alpha_1 \Rightarrow \alpha_1' + \alpha_2 < 1\ (\alpha_1 + \alpha_2 = 1) \Rightarrow$

$$P_1' = (1 - \alpha_1)(p_1 - c_1)(1 - p_1) < P_1 = \alpha_2 \left(p_m^2 - c_2\right)\left(1 - p_m^2\right) \tag{16}$$

ii) $\alpha_1' > \alpha_1 \Rightarrow \alpha_1' + \alpha_2 > 1\ (\alpha_1 + \alpha_2 = 1) \Rightarrow$

$$P_1'' = \alpha_2 \left(p_m^2 - c_2\right)\left(1 - p_m^2\right) = P_1 = \alpha_2 \left(p_m^2 - c_2\right)\left(1 - p_m^2\right) \tag{17}$$

(16) and (17) show that firm 1 has no interest to deviate.

Now, let us check for firm 2. Suppose that:

i) $\alpha_2' < \alpha_2 \Rightarrow \alpha_1 + \alpha_2' < 1\ (\alpha_1 + \alpha_2 = 1) \Rightarrow$

$$P_2' = \alpha_1 (p_1 - c_1)(1 - p_1) < P_2 = \alpha_1 \left(p_m^2 - c_2\right)\left(1 - p_m^2\right) \tag{18}$$

ii) $\alpha_2' > \alpha_2 \Rightarrow \alpha_2' + \alpha_1 > 1\ (\alpha_1 + \alpha_2 = 1) \Rightarrow$

$$P_2'' = (1 - \alpha_2')\left(p_m^2 - c_2\right)\left(1 - p_m^2\right) < P_2 = \alpha_1 \left(p_m^2 - c_2\right)\left(1 - p_m^2\right) \tag{19}$$

(18) and (19) show that firm 2 has no interest to deviate.

Conclusion: The strategies $(\alpha_1, p_1(\alpha_1, \alpha_2))$, $(\alpha_1, p_1(\alpha_1, \alpha_2))$ s.t. *i)* and *ii)* are satisfied, are SPNE$_a$ of the game.

The second part of the proposition is straightforward. We all know the common result of the Bertrand paradox where both prices $\left(p_i^b\right)$ are equal to marginal costs and profits $\left(P_i^b\right)$ are zero[10]. Hence, the difference between the both profits is:

$$P_i - P_i^b = \alpha_j \left(p_m^2 - c_2\right)\left(1 - p_m^2\right) - 0 = \alpha_j \left(p_m^2 - c_2\right)\left(1 - p_m^2\right)$$

Conclusion: If $\alpha_j > 0$, then firm i's profits in the SPNE are $\alpha_j \left(p_m^2 - c_2\right)\left(1 - p_m^2\right)$ higher than in the case where $\alpha_1 = \alpha_2 = 0$. ■

4 The general model

We consider n firms indexed by $i = 1, 2, ...n$ in a homogeneous market. We suppose that each firm incurs a cost c per unit of production[11]. The market demand function is $q = D(p) = 1 - p$. We assume that firms do not have capacity constraints and always supply the demand they face. Therefore, the profit function of firm i is:

$$\Pi_i = \begin{cases} (p_i - c)q_i & if \quad p_i < p_j \\ \frac{1}{n}(p_i - c)q_i & if \quad p_i = p_j \\ 0 & \quad\quad otherwise \end{cases} \qquad i = 1, ...n \ (i \neq j)$$

where q_i is the quantity demanded faced by firm i.

Now, let us introduce a grain of novelty in the basic Bertrand model. Let $\beta_{i1}, \beta_{i2}, ..\beta_{i\ i-1}, \beta_{i\ i+1}, ..\beta_{in}$ (resp. $\beta_{j1}, \beta_{j2}, ..\beta_{j\ j-1}, \beta_{j\ j+1}, ..\beta_{jn}$) denote the part of the profit that firm i (resp. firm j) wants to share with firms $j = 1, 2, ...n$ $(j \neq i)$ (resp. firms $i = 1, 2, ...n$ $(i \neq j)$). We suppose that $\beta_{ij}, \beta_{ji} \in]0, 1[$. Consequently, we can write the *new profit function* $P_i(p_i(..,..), p_j(..,..))$ of each firm as:

$$P_i = (1 - \textstyle\sum_{j \neq i} \beta_{ij})\Pi_i(p_i(..,..), p_j(..,..) + \textstyle\sum_{j \neq j} \beta_{ji}\Pi_j(p_i(..,..), p_j(..,..))$$

We consider a two-stage game whose sequences are thus defined. In the first stage of the game, firm i chooses $\beta_{i1}, \beta_{i2}, ..\beta_{i\ i-1}, \beta_{i\ i+1}, ..\beta_{in}$. In the second stage of the game, firm i select p_i.

[10] To avoid confusion with our model, we denote by p_i^b (resp. P_i^b) the prices (resp. the profits) in the basic Betrand model.

[11] In the next section, we will relax this assumption by allowing firms to have different marginal costs.

In the *first stage of the game*, for A and B firms simultaneously solve[12]:

$$Max_A \quad P_i = (1 - \sum_{j \neq i} \beta_{ij})\Pi_i + \sum_{j \neq i} \beta_{ji}\Pi_j$$
$$Max_B \quad P_j = (1 - \sum_{i \neq j} \beta_{ji})\Pi_j + \sum_{i \neq j} \beta_{ij}\Pi_i$$

In the *second stage of game*, for p_i and p_j firms simultaneously solve:

$$Max_{p_i} \quad P_i = (1 - \sum_{j \neq i} \beta_{ij})\Pi_i + \sum_{j \neq i} \beta_{ji}\Pi_j$$
$$Max_{p_j} \quad P_j = (1 - \sum_{i \neq j} \beta_{ji})\Pi_j + \sum_{i \neq j} \beta_{ij}\Pi_i$$

4.1 Solving the second-stage of the game

To find the subgame perfect Nash equilibrium (SPNE), we begin by solving subgames in the second-stage. Recall that, in the second stage, firms are looking for prices that maximize their profits.

Proposition 9 *If $\sum_{j \neq i} \beta_{ij} + \frac{1}{n-1} \sum_{j \neq i} \beta_{ji}) = 1$, then any prices $(p_1, p_2, ...p_n)$ such that $c \leq p_1 = p_2 = ...p_n \leq p_m$ are NE$_a$ in the second stage of the game*

Proof. $(p_1, p_2, ...p_n)$ such that $c \leq p_1 = p_2 = ...p_n \leq p_m$ are NE$_a$ if and only if no firm wants to deviate from those prices by fixing a price p_i' above or below. In fact:

$c \leq p_1 = p_2 = ...p_n \leq p_m \Rightarrow \Pi_i = \Pi_j > 0$

$\Pi_i = \frac{1}{n} (p_i - c) (1 - p_i) = \frac{1}{n} (p - c) (1 - p)$

$\Pi_j = \frac{1}{n} (p_j - c) (1 - p_j) = \frac{1}{n} (p - c) (1 - p)$

$P_i = \frac{1}{n} \left(1 - \sum_{j \neq i} \beta_{ij}\right) \Pi_i + \sum_{j \neq i} \beta_{ji}\Pi_j$

$P_i = \frac{1}{n} \left(1 - \sum_{j \neq i} \beta_{ij} + \sum_{j \neq i} \beta_{ji}\right) (p - c) (1 - p)$

[12] For writing simplication reasons, we denote $A = \beta_{i1}, \beta_{i2}, ..\beta_{i\ i-1}, \beta_{i\ i+1}, ..\beta_{in}$ and $B = \beta_{j1}, \beta_{j2}, ..\beta_{j\ j-1}, \beta_{j\ j+1}, ..\beta_{jn}$

$P_j = \frac{1}{n}\left(1 - \sum_{i \neq j} \beta_{ji} + \sum_{i \neq j} \beta_{ij}\right)(p-c)(1-p)$

Suppose that:

i) $\exists!\ i : p_i = p$ and $\forall\ j \neq i\ p_j > p\ (p_i = p_j - \varepsilon, \varepsilon > 0) \Longleftrightarrow$

$\Pi_i = (1-p_i)(p_i - c) > 0$ and $\Pi_j = 0$

$P_i' = \left(1 - \sum_{j \neq i} \beta_{ij}\right)\Pi_i = \left(1 - \sum_{j \neq i} \beta_{ij}\right)(1-p_i)(p_i - c)$

If $p_i \leq p_m$(monopolistic price), then $p_i = p - \varepsilon$.

For ε very small[13], $P_i' \simeq \left(1 - \sum_{j \neq i} \beta_{ij}\right)(1-p)(p-c) \leq P_i \Leftrightarrow$

$\left(1 - \sum_{j \neq i} \beta_{ij}\right) \leq \frac{1}{n}\left(1 - \sum_{j \neq i} \beta_{ij} + \sum_{j \neq i} \beta_{ji}\right)$ or

$$(n-1)\sum_{j \neq i} \beta_{ij} + \sum_{j \neq i} \beta_{ji} \geq n-1 \tag{20}$$

ii) $\exists!\ i : p_i = p$ and $\forall\ j \neq i\ p_j < p \Longleftrightarrow \Pi_j = \frac{1}{n-1}(1-p_j)(p_j - c) > 0$ & $\Pi_i = 0$

$P_i'' = \sum_{j \neq i} \beta_{ji}\Pi_j = \frac{1}{n-1}\sum_{j \neq i} \beta_{ji}(1-p_j)(p_j - c)$

$P_i'' = \frac{1}{n-1}\sum_{j \neq i} \beta_{ji}(1-p)(p-c) \leq P_i \Leftrightarrow$

$\frac{1}{n-1}\sum_{j \neq i} \beta_{ji} \leq \frac{1}{n}\left(1 - \sum_{j \neq i} \beta_{ij} + \sum_{j \neq i} \beta_{ji}\right)$ or

$$(n-1)\sum_{j \neq i} \beta_{ij} + \sum_{j \neq i} \beta_{ji} \leq n-1 \tag{21}$$

Equations (20) and (21) represent the non-deviation conditions and are both satisfied when $\sum_{j \neq i} \beta_{ij} + \frac{1}{n-1}\sum_{j \neq i} \beta_{ji} = 1$

Conclusion: if $\sum_{j \neq i} \beta_{ij} + \frac{1}{n-1}\sum_{j \neq i} \beta_{ji} = 1$, any prices $(p_1, p_2, ...p_n)$ such that $c \leq p_1 = p_2 = ...p_n \leq p_m$ are NE_a in the second-stage of the game. ■

[13] There is no reason for not to suppose that ε is very small. For instance, firms need to decrease or increase just slightly to get or to lose the entire market.

Proposition 10 *If* $\sum_{j\neq i}\beta_{ij}+\frac{1}{n-1}\sum_{j\neq i}\beta_{ji}>1$, *then any prices* $(p_1,p_2,...p_n)$ *such that* $p_j=p_m$ $\forall j\neq i$ *and* $p_i>p_m$ *for some* i, *are* NE_a *in the second stage of the game*

Proof. $(p_1,p_2,...p_n)$ such that $p_j=p_m$ $\forall j\neq i$ and $p_i>p_m$ for some i, are NE_a if and only if no firm has interest to deviate from those prices by fixing a price p_i' above or below.

$\exists!\ i : p_i>p_m\ \&\ \forall\ j\neq i\ p_j=p_m\Rightarrow\Pi_i=0\ \&\ \Pi_j=\frac{1}{n-1}\left(p_j-c\right)\left(1-p_j\right)>0$

$P_i=\sum_{j\neq i}\beta_{ji}\Pi_j=\frac{1}{n-1}\sum_{j\neq i}\beta_{ji}\left(p_j-c\right)\left(1-p_j\right)$

$P_j=\left(1-\sum_{i\neq j}\beta_{ji}\right)\Pi_j=\frac{1}{n-1}\left(1-\sum_{i\neq j}\beta_{ji}\right)\left(p_j-c\right)\left(1-p_j\right)$

Suppose that:

i) $\exists!\ i : p_i<p_m\ \&\ \forall\ j\neq i\ p_j=p_m\Longleftrightarrow\Pi_i=\left(1-p_i\right)\left(p_i-c\right)\ \&\ \Pi_j=0$

$P_i'=\left(1-\sum_{j\neq i}\beta_{ij}\right)\Pi_i=\left(1-\sum_{j\neq i}\beta_{ij}\right)\left(1-p_j+\varepsilon\right)\left(p_j-\varepsilon-c\right)$

For ε very small, $P_i'\simeq\left(1-\sum_{j\neq i}\beta_{ij}\right)\left(1-p_j\right)\left(p_j-c\right)<P_i\Leftrightarrow$

$\left(1-\sum_{j\neq i}\beta_{ij}\right)<\frac{1}{n-1}\sum_{j\neq i}\beta_{ji}$ or

$$\sum_{j\neq i}\beta_{ij}+\frac{1}{n-1}\sum_{j\neq i}\beta_{ji}>1 \tag{22}$$

Equation (22) represents the non-deviation condition for firm i.

Conclusion: if $\sum_{j\neq i}\beta_{ij}+\frac{1}{n-1}\sum_{j\neq i}\beta_{ji}>1$, any prices $(p_1,p_2,...p_n)$ such that $p_j=p_m$ $\forall j\neq i$ and $p_i>p_m$ for some i, are NE_a in the second-stage of the game. ■

Proposition 11 *If* $\sum_{j\neq i}\beta_{ij}+\frac{1}{n-1}\sum_{j\neq i}\beta_{ji}<1$, *then any prices* $(p_1,..p_i,..p_n)$ *such that* $p_1=...=p_i=...p_n=c$ *are* NE_a *in the second stage of the game*

Proof. $(p_1, ..p_i, ..p_n)$ such that $p_1 = ... = p_i = ...p_n = c$ are NE$_a$ if and only if no firm has interest to deviate from those prices to fix a price p_i' above or below.

$p_1 = ... = p_i = ...p_n = c \Rightarrow \Pi_i = 0$ and $\Pi_j = 0$

$P_i = \left(1 - \sum_{j \neq i} \beta_{ij}\right) \Pi_i + \sum_{j \neq i} \beta_{ji} \Pi_j = 0$

$P_j = \left(1 - \sum_{i \neq j} \beta_{ji}\right) \Pi_j + \sum_{i \neq j} \beta_{ij} \Pi_i = 0$

Suppose that:

i) $\exists!\ i : p_i = p$ and $\forall\ j \neq i\ p_j > p\ (p_i < p_j) \Rightarrow \Pi_i = (1 - p_i)(p_i - c) < 0$ and $\Pi_j = 0$

$P_i' = \left(1 - \sum_{j \neq i} \beta_{ij}\right) \Pi_i < 0$

$P_i' < P_i = 0 \Rightarrow$Firm i has no interest by fixing a price below p_j

ii) $\exists!\ i : p_i = p$ and $\forall\ j \neq i\ p_j < p\ (p_i > p_j) \iff \Pi_j = (1 - p_j)(p_j - c) = 0$ and $\Pi_i = 0$ (firm i does not produce)

$P_i'' = \sum_{j \neq i} \beta_{ji} \Pi_j = P_1 = 0 \Rightarrow$Firm i has no interest by fixing a price above p_j

Conclusion: if $\sum_{j \neq i} \beta_{ij} + \frac{1}{n-1} \sum_{j \neq i} \beta_{ji} < 1$, any prices $(p_1, ..p_i, ..p_n)$ such that $p_1 = ... = p_i = ...p_n = c$ are NE$_a$ in the second-stage of the game. ■

The second-stage being entirely solved and NE being found, we can thus move to the first-stage of the game in order to find SPNE

4.2 Solving the first-stage of the game

In the first-stage of the game, firms choose the β_{ij} or β_{ji} optimal maximizing their profits to share with their rivals.

Solving backwards, we have solved the second-stage of the game in the previous section and have found NE$_a$ in prices summarized below[14]:

[14] One can easily check that, if $\sum_{j \neq i} \beta_{ij} + \frac{1}{n-1} \sum_{j \neq i} \beta_{ji} = 1$ and $\sum_{i \neq j} \beta_{ji} + \frac{1}{n-1} \sum_{i \neq j} \beta_{ij} < 1$, then $\frac{1}{n}\left(1 - \sum_{j \neq i} \beta_{ij} + \sum_{j \neq i} \beta_{ji}\right) = \frac{1}{n-1} \sum_{j \neq i} \beta_{ji}$ and $\frac{1}{n}\left(1 - \sum_{i \neq j} \beta_{ji} + \sum_{i \neq j} \beta_{ij}\right) = \frac{1}{n-1} \sum_{i \neq j} \beta_{ij}$

$i)$ $(p_1,..p_i,..p_n) : p_1 = ... = p_i = ...p_n = c$ if $\sum_{j\neq i}\beta_{ij} + \frac{1}{n-1}\sum_{j\neq i}\beta_{ji} < 1$ with:

$$\begin{cases} P_i = 0 \\ P_j = 0 \end{cases}$$

$ii)$ $(p_1,..p_i,..p_n) : c \leq p_1 = ... = p_i = ...p_n \leq p_m$ if $\sum_{j\neq i}\beta_{ij} + \frac{1}{n-1}\sum_{j\neq i}\beta_{ji} = 1$ with:

$$\begin{cases} P_i = \frac{1}{n-1}\sum_{j\neq i}\beta_{ji}(p-c)(1-p) \\ P_j = \frac{1}{n-1}\sum_{i\neq j}\beta_{ij}(p-c)(1-p) \end{cases}$$

$iii)$ $(p_1,..p_i,..p_n) : \exists!\ i : p_i > p_m\ \&\ \forall\ j \neq i\ p_j = p_m$ if $\sum_{j\neq i}\beta_{ij} + \frac{1}{n-1}\sum_{j\neq i}\beta_{ji} > 1$ with:

$$\begin{cases} P_i = \frac{1}{n-1}\sum_{j\neq i}\beta_{ji}(p_m-c)(1-p_m) \\ P_j = \frac{1}{n-1}\left(1-\sum_{j\neq i}\beta_{ji}\right)(p_m-c)(1-p_m) \end{cases}$$

Now, in the current section, we draw our attention to the first-stage of the game searching for SPNE in β_{ij} and β_{ji}.

Proposition 12 *The strategies* $\left(\beta_{12},..,\beta_{1j(j\neq 1)},..\beta_{1n},\ p_1\left(..,\beta_{1j},..,..,\beta_{j1},..\right)\right),.....,$ $\left(\beta_{n1},..,\beta_{nj(j\neq n)},..\beta_{n1},\ p_n\left(..,\beta_{nj},..,..,\beta_{jn},..\right)\right)$ *s.t.:*

i) $\beta_{ij},\beta_{ji} \in]0,1[\ \ \&\ \ \sum_{j\neq i}\beta_{ij} + \frac{1}{n-1}\sum_{j\neq i}\beta_{ji} = 1$

ii) $\begin{cases} p_1 = ... = p_i = ...p_n = c\ if\ \ \sum_{j\neq i}\beta_{ij} + \frac{1}{n-1}\sum_{j\neq i}\beta_{ji} < 1 \\ p_1 = ... = p_i = ...p_n = p_m\ if\ \ \sum_{j\neq i}\beta_{ij} + \frac{1}{n-1}\sum_{j\neq i}\beta_{ji} = 1 \\ \exists! i : p_i > p_m\ \&\ \forall j \neq i\ p_j = p_m\ if\ \ \sum_{j\neq i}\beta_{ij} + \frac{1}{n-1}\sum_{j\neq i}\beta_{ji} > 1 \end{cases}$

are SPNE$_a$ of the game.

Furthermore, if $\beta_{ji} > 0$*, then firm i's profits in the SNPE$_a$ are* $\frac{1}{n-1}\sum_{j\neq i}\beta_{ji}(p_m-c)(1-p_m)$ *higher than in the case where* $\alpha_1 = \alpha_2 = 0$*.*

Proof. Let us show the first part of the proposition

The strategies $\left(\beta_{12},..,\beta_{1j(j\neq 1)},..\beta_{1n},\ p_1\left(..,\beta_{1j},..,..,\beta_{j1},..\right)\right),.....,$ $\left(\beta_{n1},..,\beta_{nj(j\neq n)},..\beta_{n1},\ p_n\left(..,\beta_{nj},..,..,\beta_{jn},..\right)\right)$ s.t. $i)$ and $ii)$ are satisfied, are SPNE$_a$ if and only if no firm has interest to deviate from those prices by choosing a β'_{ij} or β'_{ji} above or below. Because of the multiplicity of β'_{ij} and β'_{ji}, we investigate separately the deviation for each firm.

Let us check first for firm i. Suppose that:

$i)$ $\beta'_{ij} < \beta_{ij} \Rightarrow \sum_{j\neq i} \beta'_{ij} + \frac{1}{n-1} \sum_{j\neq i} \beta_{ji} < 1 \Rightarrow$

$$P'_i = 0 < P_i = \frac{1}{n-1} \sum_{j\neq i} \beta_{ji} (p_m - c)(1 - p_m) \tag{23}$$

$ii)$ $\beta'_{ij} > \beta_{ij} \Rightarrow \sum_{j\neq i} \beta'_{ij} + \frac{1}{n-1} \sum_{j\neq i} \beta_{ji} > 1 \Rightarrow$

$$P''_i = \frac{1}{n-1} \sum_{j\neq i} \beta_{ji} (p_m - c)(1 - p_m) = P_i \tag{24}$$

(23) and (24) show that firm i has no interest to deviate.

Now, let us check for firm j. Suppose that:

$i)$ $\beta'_{ji} < \beta_{ji} \Rightarrow \sum_{j\neq i} \beta_{ij} + \frac{1}{n-1} \sum_{j\neq i} \beta'_{ji} < 1 \Rightarrow$

$$P'_j = 0 < P_j = \frac{1}{n-1} \sum_{j\neq i} \beta_{ij} (p_m - c)(1 - p_m) \tag{25}$$

$ii)$ $\beta'_{ji} > \beta_{ji} \Rightarrow \sum_{j\neq i} \beta_{ij} + \frac{1}{n-1} \sum_{j\neq i} \beta'_{ji} > 1 \Rightarrow$

$$P''_j = \frac{1}{n-1} \left(1 - \sum_{j\neq i} \beta_{ji}\right) (p_m - c)(1 - p_m) < P_j \tag{26}$$

(25) and (26) show that firm j has no interest to deviate.

Finally, we conclude that the strategies $\left(\beta_{12}, .., \beta_{1j(j\neq 1)}, ..\beta_{1n}, p_1 \left(.., \beta_{1j}, ..,.., \beta_{j1}, ..\right)\right)$, ..., $\left(\beta_{n1}, .., \beta_{nj(j\neq n)}, ..\beta_{n1}, p_n \left(.., \beta_{nj}, ..,.., \beta_{jn}, ..\right)\right)$ s.t. $i)$ and $ii)$ are satisfied, are SPNE$_a$ of the game.

The second part of the proposition is straightforward. We all know the common result of the Bertrand paradox where both prices $\left(p_i^b\right)$ are equal to marginal costs and profits $\left(P_i^b\right)$ are zero[15]. Hence, the difference between the both profits is:

$$P_i - P_i^b = \frac{1}{n-1} \sum_{j\neq i} \beta_{ji} (p_m - c)(1 - p_m) - 0 = \frac{1}{n-1} \sum_{j\neq i} \beta_{ji} (p_m - c)(1 - p_m)$$

Conclusion: If $\alpha_j > 0$, then firm i's profits in the SPNE$_a$ are $\frac{1}{n-1} \sum_{j\neq i} \beta_{ji} (p_m - c)(1 - p_m)$ higher than in the case where $\alpha_1 = \alpha_2 = 0$.

■

[15] To avoid confusion with our model, we denote by p_i^b (resp. P_i^b) the prices (resp. the profits) in the basic Betrand model.

5 The general model modified

We consider the same model as before except that we allow firms to have *different marginal costs*. We still consider n firms indexed by $i = 1, 2, ...n$ in a homogeneous market. *Here*, we suppose that each firm incurs a cost c_i ($c_1 < c_2 < ... < c_n$) per unit of production. Therefore, the profit function of firm i becomes:

$$\Pi_i = \begin{cases} (p_i - c_i)q_i & if \quad p_i < p_j \\ \frac{1}{n}(p_i - c_i)q_i & if \quad p_i = p_j \\ 0 & \quad\quad otherwise \end{cases} \quad i = 1, ...n \ (i \neq j)$$

where q_i is the quantity demanded faced by firm i.

Now, let us introduce a grain of novelty in the basic Bertrand model. Let $\beta_{i1}, \beta_{i2}, ..\beta_{i\ i-1}, \beta_{i\ i+1}, ..\beta_{in}$ (resp. $\beta_{j1}, \beta_{j2}, ..\beta_{j\ j-1}, \beta_{j\ j+1}, ..\beta_{jn}$) denote the part of the profit that firm i (resp. firm j) wants to share with firms $j = 1, 2, ...n$ $(j \neq i)$ (resp. firms $i = 1, 2, ...n$ $(i \neq j)$). We suppose that $\beta_{ij}, \beta_{ji} \in]0, 1[$. Consequently, we can write the *new profit function* $P_i(p_i(..,..), p_j(..,..))$ (hereafter P_i) of each firm as:

$$P_i = (1 - \textstyle\sum_{j\neq i} \beta_{ij})\Pi_i(p_i(..,..), p_j(..,..) + \sum_{j\neq j} \beta_{ji}\Pi_j(p_i(..,..), p_j(..,..))$$

We consider a two-stage game whose sequences are thus defined. In the first stage of the game, firm i chooses $\beta_{i1}, \beta_{i2}, ..\beta_{i\ i-1}, \beta_{i\ i+1}, ..\beta_{in}$. In the second stage of the game, firm i selects p_i.

In the *first stage of the game*, for A and B firms simultaneously solve[16]:

$$Max_A \quad P_i = (1 - \textstyle\sum_{j\neq i} \beta_{ij})\Pi_i + \sum_{j\neq i} \beta_{ji}\Pi_j$$

$$Max_B \quad P_j = (1 - \textstyle\sum_{i\neq j} \beta_{ji})\Pi_j + \sum_{i\neq j} \beta_{ij}\Pi_i$$

In the *second stage of game*, for p_i and p_j firms simultaneously solve:

$$Max_{p_i} \quad P_i = (1 - \textstyle\sum_{j\neq i} \beta_{ij})\Pi_i + \sum_{j\neq i} \beta_{ji}\Pi_j$$

$$Max_{p_j} \quad P_j = (1 - \textstyle\sum_{i\neq j} \beta_{ji})\Pi_j + \sum_{i\neq j} \beta_{ij}\Pi_i$$

[16] For writing simplication reasons, we denote $A = \beta_{i1}, \beta_{i2}, ..\beta_{i\ i-1}, \beta_{i\ i+1}, ..\beta_{in}$ and $B = \beta_{j1}, \beta_{j2}, ..\beta_{j\ j-1}, \beta_{j\ j+1}, ..\beta_{jn}$

5.1 Solving the second-stage of the game

To find the subgame perfect Nash equilibrium (SPNE), we begin by solving subgames in the second-stage. Recall that, in the second stage, firms are looking for prices that maximize their profits.

Proposition 13 *If* $\sum_{j\neq i}\beta_{ij}+\sum_{j\neq i}\beta_{ji})=1$*, then any prices* $(p_1,p_2,...p_n)$ *such that* $c_n\leq p_1=p_2=...p_n\leq p_m^n$ *(firm n monopolistic price) are* NE_a *in the second stage of the game*

Proof. $(p_1,p_2,...p_n)$ such that $c_n\leq p_1=p_2=...p_n\leq p_m^n$ are NE_a if and only if no firm wants to deviate from those prices by fixing a price p_i' above or below. In fact:

$$c_n\leq p_1=p_2=...p_n\leq p_m\Rightarrow\Pi_i=\Pi_j>0$$

$$\Pi_i=\tfrac{1}{n}\left(p_i-c_i\right)\left(1-p_i\right)=\tfrac{1}{n}\left(p-c_i\right)\left(1-p\right)$$

$$\Pi_j=\tfrac{1}{n}\left(p_j-c_j\right)\left(1-p_j\right)=\tfrac{1}{n}\left(p-c_j\right)\left(1-p\right)$$

$$P_i=\tfrac{1}{n}\left(1-\sum\nolimits_{j\neq i}\beta_{ij}\right)\Pi_i+\sum\nolimits_{j\neq i}\beta_{ji}\Pi_j$$

$$P_i=\tfrac{1}{n}\left(1-p\right)\left[\left(1-\sum\nolimits_{j\neq i}\beta_{ij}\right)\left(p-c_i\right)+\sum\nolimits_{j\neq i}\beta_{ji}\left(p-c_j\right)\right]$$

$$P_j=\tfrac{1}{n}\left(1-p\right)\left[\left(1-\sum\nolimits_{i\neq j}\beta_{ji}\right)\left(p-c_j\right)+\sum\nolimits_{i\neq j}\beta_{ij}\left(p-c_i\right)\right]$$

Since firms are different, we shall study separately the deviation. Let us check first for firm i. Suppose that:

i) $\exists!\ i: p_i=p\ \&\ \forall\ j\neq i,\ p_j>p\ (p_i=p_j-\varepsilon,\varepsilon>0)\iff\Pi_i=(1-p_i)(p_i-c_i)>0$ and $\Pi_j=0$

$$P_i'=\left(1-\sum\nolimits_{j\neq i}\beta_{ij}\right)\Pi_i=\left(1-\sum\nolimits_{j\neq i}\beta_{ij}\right)\left(1-p_i\right)\left(p_i-c_i\right)$$

If $p_i\leq p_m$(monopolistic price), then $p_i=p-\varepsilon$.

For ε very small[17], $P_i'\simeq\left(1-\sum_{j\neq i}\beta_{ij}\right)(1-p)(p-c_i)\leq P_i\Leftrightarrow$

[17] There is no reason for not to suppose that ε is very small. For instance, firms need to decrease or increase just slightly to get or to lose the entire market.

$$\left(1-\sum_{j\neq i}\beta_{ij}\right)(p-c_i)\leq \tfrac{1}{n}\left[\left(1-\sum_{j\neq i}\beta_{ij}\right)(p-c_i)+\sum_{j\neq i}\beta_{ji}(p-c_j)\right]$$

or

$$(n-1)\,\frac{1-\sum_{j\neq i}\beta_{ij}}{\sum_{j\neq i}\beta_{ji}}\leq\frac{p-c_j}{p-c_i} \tag{27}$$

ii) $\exists!\ i: p_i = p\ \&\ \forall\ j\neq i,\ p_j < p\ (p_i > p_j) \Longleftrightarrow$

$\Pi_j = \frac{1}{n-1}(1-p_j)(p_j-c_j) > 0\ \&\ \Pi_i = 0$

$P_i'' = \sum_{j\neq i}\beta_{ji}\Pi_j = \frac{1}{n-1}\sum_{j\neq i}\beta_{ji}(1-p_j)(p_j-c_j)$

$P_i'' = \frac{1}{n-1}\sum_{j\neq i}\beta_{ji}(1-p)(p-c_j)\leq P_i \Leftrightarrow$

$\frac{1}{n-1}\sum_{j\neq i}\beta_{ji}\leq\frac{1}{n}\left(1-\sum_{j\neq i}\beta_{ij}+\sum_{j\neq i}\beta_{ji}\right)$ or

$$\frac{p-c_j}{p-c_i}\leq(n-1)\,\frac{1-\sum_{j\neq i}\beta_{ij}}{\sum_{j\neq i}\beta_{ji}} \tag{28}$$

Equations (27) and (28) represent the non-deviation conditions and are both satisfied when $(n-1)\,\frac{1-\sum_{j\neq i}\beta_{ij}}{\sum_{j\neq i}\beta_{ji}}=\frac{p-c_j}{p-c_i}$

Let us check now for firm j. Suppose that:

i) $\forall\ j,\ \exists! i: p_i = p$ and $\forall\ j\neq i,\ p_j < p\ (p_i = p_j-\varepsilon,\varepsilon>0) \Longleftrightarrow \Pi_j = \frac{1}{n-1}(1-p_j)(p_j-c_j) > 0$ and $\Pi_j = 0$

$P_j' = \left(1-\sum_{j\neq i}\beta_{ji}\right)\Pi_j = \frac{1}{n-1}\left(1-\sum_{j\neq i}\beta_{ji}\right)(1-p_j)(p_j-c_j)$

If $p_j\leq p_m$(monopolistic price), then $p_j = p-\varepsilon$.

For ε very small[18], $P_j'\simeq\frac{1}{n-1}\left(1-\sum_{j\neq i}\beta_{ji}\right)(1-p)(p-c_j)\leq P_j\Leftrightarrow$

$$\tfrac{1}{n-1}\left(1-\sum_{i\neq j}\beta_{ji}\right)(p-c_j)\leq\tfrac{1}{n}\left[\left(1-\sum_{i\neq j}\beta_{ji}\right)(p-c_j)+\sum_{i\neq j}\beta_{ij}(p-c_i)\right]$$

or

$$\frac{p-c_j}{p-c_i}\leq(n-1)\,\frac{\sum_{i\neq j}\beta_{ij}}{1-\sum_{i\neq j}\beta_{ji}} \tag{29}$$

ii) $\forall\ j,\ \exists! i: p_i = p\ \&\ \forall\ j\neq i,\ p_j > p \Longleftrightarrow \Pi_i = (1-p_i)(p_i-c_i) > 0\ \&\ \Pi_i = 0$

[18] There is no reason for not to suppose that ε is very small. For instance, firms need to decrease or increase just slightly to get or to lose the entire market.

$P_j'' = \sum_{i \neq j} \beta_{ij} \Pi_i = \sum_{i \neq j} \beta_{ij} (1 - p_i)(p_i - c_i) \simeq \sum_{i \neq j} \beta_{ij} (1 - p)(p - c_i) \leq P_i \Leftrightarrow$

$\sum_{i \neq j} \beta_{ij} (p - c_i) \leq \frac{1}{n} \left[\left(1 - \sum_{i \neq j} \beta_{ji}\right)(p - c_j) + \sum_{i \neq j} \beta_{ij} (p - c_i) \right]$ or

$$(n-1) \frac{\sum_{i \neq j} \beta_{ij}}{1 - \sum_{i \neq j} \beta_{ji}} \leq \frac{p - c_j}{p - c_i} \tag{30}$$

Equations (29) and (30) represent the non-deviation conditions for firm j and are both satisfied when $(n-1) \frac{\sum_{i \neq j} \beta_{ij}}{1 - \sum_{i \neq j} \beta_{ji}} = \frac{p - c_j}{p - c_i}$

Equations $(27) - (30)$ represent the non-deviation conditions for both firms and are all satisfied when $\sum_{j \neq i} \beta_{ij} + \sum_{j \neq i} \beta_{ji} = 1$

Conclusion: if $\sum_{j \neq i} \beta_{ij} + \sum_{j \neq i} \beta_{ji} = 1$, any prices $(p_1, p_2, ...p_n)$ such that $c_n \leq p_1 = p_2 = ...p_n \leq p_m$ are NE in the second-stage of the game. ■

Proposition 14 *If $\sum_{j \neq i} \beta_{ij} + \sum_{j \neq i} \beta_{ji} > 1$, then any prices $(p_1, p_2, ...p_n)$ such that $p_j = p_m^n$ $\forall j \neq i$ and $p_i > p_m^n$ for some i, are NE$_a$ in the second stage of the game*

Proof. $(p_1, p_2, ...p_n)$ such that $p_j = p_m$ $\forall j \neq i$ and $p_i > p_m$ for some i, are NE$_a$ if and only if no firm has interest to deviate from those prices by fixing a price p_i' above or below.

$\exists!\ i : p_i > p_m$ and $\forall\ j \neq i\ p_j = p_m \Rightarrow \Pi_i = 0$ & $\Pi_j = \frac{1}{n-1} (p_j - c_j)(1 - p_j) > 0$

$P_i = \sum_{j \neq i} \beta_{ji} \Pi_j = \frac{1}{n-1} \sum_{j \neq i} \beta_{ji} (p_j - c_j)(1 - p_j)$

$P_j = \left(1 - \sum_{j \neq i} \beta_{ji}\right) \Pi_j = \frac{1}{n-1} \left(1 - \sum_{j \neq i} \beta_{ji}\right) (p_j - c_j)(1 - p_j)$

Since firms are different, we shall study separately the deviation. Let us check first for firm i. Suppose that:

i) $\exists!\ i : p_i < p_m$ and $\forall\ j \neq i\ p_j = p_m \iff \Pi_i = (1 - p_i)(p_i - c_i)$ and $\Pi_j = 0$

$P_i' = \left(1 - \sum_{j \neq i} \beta_{ij}\right) \Pi_i = \left(1 - \sum_{j \neq i} \beta_{ij}\right) (1 - p_j + \varepsilon)(p_j - c_i - \varepsilon)$

For ε very small, $P_i' \simeq \left(1 - \sum_{j \neq i} \beta_{ij}\right)(1 - p_j)(p_j - c_i) < P_i \Leftrightarrow$

$\left(1 - \sum_{j \neq i} \beta_{ij}\right)(p_j - c_i) < \frac{1}{n-1} \sum_{j \neq i} \beta_{ji} (p_j - c_j)$ or

$$(n-1) \frac{1 - \sum_{j \neq i} \beta_{ij}}{\sum_{j \neq i} \beta_{ji}} \leq \frac{p_j - c_j}{p_j - c_i} \tag{31}$$

Equation (31) represent the non-deviation condition for firm i.

Let us check now for firm j. Suppose that:

ii) $\forall\ j$, $\exists! i : p_i = p$ and $\forall\ j \neq i$, $p_j > p \Longleftrightarrow \Pi_i = (1 - p_i)(p_i - c_i) > 0$ & $\Pi_j = 0$

$P_j'' = \sum_{j \neq i} \beta_{ij} \Pi_i = \sum_{j \neq i} \beta_{ij} (1 - p_i + \varepsilon)(p_i - c_i - \varepsilon)$

$P_j'' \simeq \sum_{j \neq i} \beta_{ij} (1 - p_j)(p_j - c_i) \leq P_j \Leftrightarrow$

$\sum_{j \neq i} \beta_{ij} (p - c_i) \leq \frac{1}{n-1} \left(1 - \sum_{j \neq i} \beta_{ji}\right)(p_j - c_j)$ or

$$(n-1) \frac{\sum_{j \neq i} \beta_{ij}}{1 - \sum_{j \neq i} \beta_{ji}} \geq \frac{p - c_j}{p - c_i} \tag{32}$$

Equation (32) represent the non-deviation condition for firm i.

Conclusion: if $\sum_{j \neq i} \beta_{ij} + \sum_{j \neq i} \beta_{ji} > 1$, any prices $(p_1, p_2, ...p_n)$ such that $p_j = p_m\ \forall j \neq i$ and $p_i > p_m$ for some i, are NE$_a$ in the second-stage of the game. ■

Proposition 15 *If $\sum_{j \neq i} \beta_{ij} + \sum_{j \neq i} \beta_{ji} < 1$, then any prices $(p_1, ..p_j, ..p_n)$ such that $p_j = c_n - \varepsilon$ $(j \neq n,\ \varepsilon > 0)$ and $p_n = c_n$ are NE$_a$ in the second stage of the game*

Proof. $p_j = c_n - \varepsilon$ $(j \neq n,\ \varepsilon > 0)$ and $p_n = c_n$ are NE$_a$ if and only if no firm has interest to deviate from those prices to fix a price p_j' above or below.

$p_j = c_n - \varepsilon$ $(j \neq n,\ \varepsilon > 0)$ and $p_n = c_n \Rightarrow \Pi_j = \frac{1}{n-1}(p_j - c_j)(1 - p_j) > 0$ and $\Pi_n = 0$

$P_j = \left(1 - \sum_{i \neq j} \beta_{ji}\right) \Pi_j = \frac{1}{n-1} \left(1 - \sum_{i \neq j} \beta_{ji}\right)(p_j - c_j)(1 - p_j)$

$P_n = \sum_{j \neq n} \beta_{jn} \Pi_j = \sum_{i \neq n} \beta_{ji} \Pi_j = \sum_{i \neq n} \beta_{ji} (p_j - c_j)(1 - p_j)$

We will study the deviation for firm j and firm n. Let us check first for firm j. Suppose that:

i) $\exists!\ n : p_n = c_n$ and $\forall\ j \neq n$, $p'_j < p_j \Rightarrow \Pi_j = \frac{1}{n-1}\left(1 - p'_j\right)\left(p'_j - c_j\right)$ and $\Pi_n = 0$

$P'_j = \left(1 - \sum_{i \neq j} \beta_{ji}\right) \Pi_j$

$P'_j < P_j = 0 \Rightarrow$Firm j has no interest by fixing a price below p_j

ii) $\exists!\ n : p_n = c_n$ and $\forall\ j \neq n$, $p''_j = c_n > p_j \iff \Pi_n = 0$ and $\Pi_j = \frac{1}{n}\left(1 - p''_j\right)\left(p''_j - c_j\right)$

$P''_j = \left(1 - \sum_{i \neq j} \beta_{ji}\right) \Pi_j < P_j \Rightarrow$Firm j has no interest by fixing a price above p_j

Let us check now for firm n. Suppose that:

i) $\exists!\ n : p_n < c_n$ and $\forall\ j \neq n$, $p_j > p_n \Rightarrow \Pi_n = (1 - p_n)(p_n - c_n) < 0$ and $\Pi_j = 0$

$P'_n = \left(1 - \sum_{n \neq j} \beta_{nj}\right) \Pi_n < 0$

$P'_n < P_n = 0 \Rightarrow$Firm n has no interest by fixing a price below p_n

ii) $\exists!\ n : p_n > c_n$ and $\forall\ j \neq n$, $p_j < p_n \iff \Pi_j = \frac{1}{n-1}(p_j - c_j)(1 - p_j)$ and $\Pi_n = 0$ (firm n does not produce)

$P''_n = \sum_{j \neq n} \beta_{jn} \Pi_j = P_n \Rightarrow$Firm n has no interest by fixing a price above p_n

Conclusion: if $\sum_{j \neq i} \beta_{ij} + \sum_{j \neq i} \beta_{ji} < 1$, any prices $(p_1, ..p_j, ..p_n)$ such that $p_j = c_n - \varepsilon$ $(j \neq n, \varepsilon > 0)$ and $p_n = c_n$ are NE$_a$ in the second-stage of the game. ■

Note that, in the last NE *firms' profits are positive even when they set price at the highest marginal cost.*

The second-stage being entirely solved and NE$_a$ being found, we can thus move to the first-stage of the game in order to find SPNE$_a$

5.2 Solving the first-stage of the game

In the first-stage of the game, firms choose the α_i optimal maximizing their profit to share with their rival.

Solving backwards, we have solved the second-stage of the game in the previous section and have found NE$_a$ in prices summarized below[19]:

i) $(p_1,..p_i,..p_n) : p_i = c_n - \varepsilon \ (i \neq n, \varepsilon > 0)$ and $p_n = c_n$ if $\sum_{j\neq i} \beta_{ij} + \sum_{j\neq i} \beta_{ji} < 1$ with:

$$\begin{cases} P_i = \frac{1}{n}\left(1 - \sum_{j\neq i} \beta_{ij}\right)(p_i - c_i)(1 - p_i) \\ P_n = \frac{1}{n}\sum_{i\neq n} \beta_{in}(p_i - c_i)(1 - p_i) \ldots\ldots. \end{cases}$$

ii) $(p_1,..p_i,..p_n) : c_n \leq p_1 = ... = p_i = ...p_n \leq p_m^n$ if $\sum_{j\neq i} \beta_{ij} + \sum_{j\neq i} \beta_{ji} = 1$ with:

$$\begin{cases} P_i \simeq \frac{2}{n}\sum_{j\neq i} \beta_{ji}(p - c_i)(1 - p) \text{ or } \frac{2}{n}\sum_{j\neq i} \beta_{ji}(p - c_j)(1 - p) \\ P_j \simeq \frac{2}{n}\sum_{i\neq j} \beta_{ij}(p - c_j)(1 - p) \text{ or } \frac{2}{n}\sum_{i\neq j} \beta_{ij}(p - c_i)(1 - p) \end{cases}$$

iii) $(p_1,..p_i,..p_n) : \exists!\ i : p_i > p_m^n \ \&\ \forall\ j \neq i\ p_j = p_m^n$ if $\sum_{j\neq i} \beta_{ij} + \sum_{j\neq i} \beta_{ji} > 1$ with:

$$\begin{cases} P_i = \frac{1}{n-1}\sum_{j\neq i} \beta_{ji}(p_m^n - c_j)(1 - p_m^n) \text{ or } \frac{1}{n-1}\left(1 - \sum_{j\neq i} \beta_{ij}\right)(p_m^n - c_i)(1 - p_m^n) \\ P_j = \frac{1}{n-1}\left(1 - \sum_{i\neq j} \beta_{ji}\right)(p_m^n - c_j)(1 - p_m^n) \text{ or } \frac{1}{n-1}\sum_{i\neq j} \beta_{ij}(p_m^n - c_i)(1 - p_m^n) \end{cases}$$

Note that in every NE, firms get positive profits and even when they set price at marginal cost. *This is the main difference with the previous general model where firms have equal marginal costs.*

Now, in the current section, we draw our attention to the first-stage of the game searching for SPNE$_a$ in β_{ij} and β_{ji}.

Proposition 16 *The strategies* $\left(\beta_{12}, .., \beta_{1j(j\neq 1)}, ..\beta_{1n},\ p_1\left(.., \beta_{1j}, ..,.., \beta_{j1}, ..\right)\right), \ldots..,$ $\left(\beta_{n1}, .., \beta_{nj(j\neq n)}, ..\beta_{n1},\ p_n\left(.., \beta_{nj}, ..,.., \beta_{jn}, ..\right)\right)$ *s.t.:*

i) $\beta_{ij}, \beta_{ji} \in]0, 1[\ \ \&\ \ \sum_{j\neq i} \beta_{ij} + \sum_{j\neq i} \beta_{ji} = 1$

$$ii)\begin{cases} (p_1,..p_i,..p_n) : p_i = c_n - \varepsilon \ \ (\varepsilon > 0)\ \ \&\ p_n = c_n \ if\ \ \sum_{j\neq i} \beta_{ij} + \sum_{j\neq i} \beta_{ji} < 1 \\ p_1 = ... = p_i = ...p_n = p_m^n \ if\ \ \sum_{j\neq i} \beta_{ij} + \sum_{j\neq i} \beta_{ji} = 1 \\ \exists! i : p_i > p_m^n \ \&\ \forall\ j \neq i\ p_j = p_m^n \ if\ \ \sum_{j\neq i} \beta_{ij} + \frac{1}{n-1}\sum_{j\neq i} \beta_{ji} > 1 \end{cases}$$

[19] One can easily check that, if $\sum_{j\neq i} \beta_{ij} + \sum_{j\neq i} \beta_{ji} = 1$, then $\frac{1}{n}\left(1 - \sum_{j\neq i} \beta_{ij} + \sum_{j\neq i} \beta_{ji}\right) = \frac{2}{n}\sum_{j\neq i} \beta_{ji}$ and $\frac{1}{n}\left(1 - \sum_{i\neq j} \beta_{ji} + \sum_{i\neq j} \beta_{ij}\right) = \frac{2}{n}\sum_{i\neq j} \beta_{ij}$

are SPNE$_a$ of the game.
Furthermore, if $\beta_{ji} > 0$, then firm i's profits in the SNPE$_a$ are $\frac{2}{n}\sum_{j\neq i}\beta_{ji}(p_m - c_i)(1-p_m)$ *higher than in the case where* $\alpha_1 = \alpha_2 = 0$.

Proof. Let us show the first part of the proposition.

The strategies $\left(\beta_{12},..,\beta_{1j(j\neq 1)},..\beta_{1n},\, p_1\left(..,\beta_{1j},..,..,\beta_{j1},..\right)\right),.....,$ $\left(\beta_{n1},..,\beta_{nj(j\neq n)},..\beta_{n1},\, p_n\left(..,\beta_{nj},..,..,\beta_{jn},..\right)\right)$ s.t. *i)* and *ii)* are satisfied, are SPNE$_a$ if and only if no firm has interest to deviate from those prices by choosing a β'_{ij} or β'_{ji} above or below. Because of the multiplicity of β'_{ij} and β'_{ji}, we investigate separately the deviation for each firm.

Let us check first for firm i. Suppose that:

i) $\beta'_{ij} < \beta_{ij} \Rightarrow \sum_{j\neq i}\beta'_{ij}+\sum_{j\neq i}\beta_{ji} < 1 \Rightarrow P'_i = \frac{1}{n}\left(1-\sum_{j\neq i}\beta'_{ij}\right)(p_i - c_i)(1-p_i)$

$$P'_i < P_i = \frac{2}{n}\sum_{i\neq j}\beta_{ji}\left(p^n_m - c_i\right)\left(1-p^n_m\right) \tag{33}$$

ii) $\beta'_{ij} > \beta_{ij} \Rightarrow \sum_{j\neq i}\beta'_{ij} + \sum_{j\neq i}\beta_{ji} > 1 \Rightarrow$

$$P''_i = \frac{1}{n-1}\left(1-\sum_{j\neq i}\beta'_{ij}\right)\left(p^n_m - c_i\right)\left(1-p^n_m\right) < P_i \tag{34}$$

(33) and (34) show that firm i has no interest to deviate.

Now, let us check for firm n. Suppose that:

i) $\beta'_{ni} < \beta_{ni} \Rightarrow \sum_{i\neq n}\beta'_{ni}+\sum_{i\neq n}\beta_{in} < 1 \Rightarrow P'_n = \frac{1}{n}\sum_{i\neq n}\beta_{in}(p_i - c_i)(1-p_i)$

$$P'_n < P_n = \frac{2}{n}\sum_{i\neq n}\beta_{in}\left(p^n_m - c_i\right)\left(1-p^n_m\right) \tag{35}$$

ii) $\beta'_{nj} > \beta_{nj} \Rightarrow \sum_{i\neq n}\beta'_{ni} + \sum_{i\neq n}\beta_{in} > 1 \Rightarrow$

$$P''_n = \frac{1}{n-1}\sum_{i\neq n}\beta_{in}\left(p_m - c_i\right)\left(1-p_m\right) < P_n \tag{36}$$

(35) and (36) show that firm j has no interest to deviate.

Finally, we conclude that the strategies $\left(\beta_{12},..,\beta_{1j(j\neq 1)},..\beta_{1n},\, p_1\left(..,\beta_{1j},..,..,\beta_{j1},..\right)\right),$

....., $\left(\beta_{n1},..,\beta_{nj(j\neq n)},..\beta_{n1},\, p_n\left(..,\beta_{nj},..,..,\beta_{jn},..\right)\right)$ s.t. *i*) and *ii*) are satisfied, are SPNE$_a$ of the game.

The second part of the proposition is straightforward. We all know the common result of the Bertrand paradox where both prices $\left(p_i^b\right)$ are equal to marginal costs and profits $\left(P_i^b\right)$ are zero[20]. Hence, the difference between the both profits is:

$P_i - P_i^b = \frac{2}{n}\sum_{j\neq i}\beta_{ji}\left(p_m - c_i\right)\left(1-p_m\right) - 0 = \frac{2}{n}\sum_{j\neq i}\beta_{ji}\left(p_m - c_i\right)\left(1-p_m\right)$

Conclusion: If $\alpha_j > 0$, then firm i's profits in the SPNE$_a$ are $\frac{2}{n}\sum_{j\neq i}\beta_{ji}\left(p_m - c_i\right)\left(1-p_m\right)$ higher than in the case where $\alpha_1 = \alpha_2 = 0$.

■

6 Conclusion

This paper has shown, through a particular strategy, that firms may be able *to set prices above the marginal costs and thus get positive profits.* This remarkable result is robust to the number of firms and to cost asymmetries.

Furthermore and more importantly, when firms' costs are different, *firms get positive profits even though they set prices at the highest marginal cost.*

Shall this new solution hint that competition between firms should not be reduced to the models of Bertrand, Cournot, Stackelberg and the like. We leave that question open for future research.

There are some dimensions along which our simple model can be enriched. For instance, a natural one is the extension of our analysis to the Cournot model. Such an extension should be straightforward at least for a linear function.

An other interesting area of investigation would be to allow firms to *invest* (rather than *sharing*) a part of their profits to a joint venture. *Profit Sharing Between Firms: An Application to Joint Ventures* (Waddle 2005c) focuses on this concern.

[20] To avoid confusion with our model, we denote by p_i^b (resp. P_i^b) the prices (resp. the profits) in the basic Betrand model.

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